Blocking Sets, Linear Groups and Transversal Designs

Aiden A. Bruen
Department of Electrical and Computer Engineering
University of Calgary, Canada

Francesco Mazzocca and *Olga Polverino*
Dipartimento di Matematica
Seconda Universitá degli Studi di Napoli, Italia

Abstract

We exploit new methods involving affine groups to determine the complete geometric structure of perspective sets in PG(2,q). Using this we then, in a few pages, give a complete characterization of those blocking sets in PG(2,q) that contain at least two Rédei lines. Our characterization is analogous to, but slightly more detailed than, the characterization obtained by Sherman in the sequence [13, 14]. Finally, we use our results to sharpen known results (see [5]) by obtaining a detailed classification of transversal designs embedded in planes.

1. Introduction

A *blocking set* in a projective plane is a set of points not containing a line that intersects every line. If Y is a blocking set in $PG(2,q)$ of size $q+\lambda$, then at most λ points of Y lie on a line. If a set of λ collinear points exists in Y then Y is said to be a *Rédei blocking set* and every line meeting Y in λ points is called a *Rédei line of* Y. This terminology arose as follows. Let Y be a Rédei blocking set and L a Rédei line for Y. Then every line through two distinct points x, y of $Y \setminus L$ intersects L in a point of Y. Therefore, as was first pointed out in [6] and going back also to unpublished work of T.G. Ostrom, $Y \setminus L$ can be viewed as the graph of a function in the affine plane $PG(2,q) \setminus L$ and $Y \cap L$ is the set of directions determined by this graph. L.Rédei in [12] obtained powerful results on the number of directions determined by the graph of a function in the affine plane $AG(2,q)$ and the techniques of Rédei can be used to obtain results on arbitrary blocking sets in $PG(2,q)$. The work of Rédei and its relevance to blocking sets was first pointed out in [6]: the results in that paper were subsequently extended by H.W.Lenstra Jr [11]. For additional references, see [3],[4],[2],[1].

In [13, 14], B.F.Sherman found a canonical representation for minimal blocking sets in $PG(2,q)$ with al least two Rédei lines. More precisely he proved the following.

Theorem 1.1 *A point set Y in $PG(2,q)$ of size $q+\lambda$ is a minimal blocking set with at least two Rédei lines if, and only if, there exist an additive subgroup W, a multiplicative subgroup M of $GF(q)$ (each element of which leaves W invariant), and a coordinatisation of $PG(2,q)$ such that*

$$Y = \{(x,-1,0) \ : \ x \in X\} \cup \{(x,0,1) \ : \ x \in X\} \cup$$

$$\{(1,0,0)\} \cup \{(w,m,1) \ : \ m \in M, \ w \in W\},$$

where $X = GF(q) \setminus \{x \in GF(q) \ : \ x = mv + w, \ m \in M, \ w \in W\}$, v being some element in $GF(q)$ not in $M \cup W$. The two sets of points characterized by X, along with $\{(1,0,0)\}$, are the sections of Y in the two λ-secants, where $\lambda = |X| + 1 = q - |M||W| + 1$.

In this paper we use quite different techniques, related to Dickson's classification of the subgroups of the affine group Σ on the line $AG(1,q)$. We are able to show that every blocking set with at least two Rédei lines can be described by a subgroup of Σ, and conversely. In this way we obtain a new geometric description of such blocking sets and we give a synthetic proof of Sherman's result. For related ideas we also refer to [10].

2. Preliminaries

Let $q = p^r$ be a prime power and $PG(2,q)$ the projective plane over the Galois field $GF(q)$. If L_1, L_2 are distinct lines in $PG(2,q)$ and x is a point not in $L_1 \cup L_2$ we denote by π_x^1 and π_x^2 the perspectivities with center x mapping L_1 onto L_2 and L_2 onto L_1, respectively. Put $L_1 \cap L_2 = c$.

Consider two perspective point sets $X_1 \subseteq L_1 \setminus c$, $X_2 \subseteq L_2 \setminus c$ and denote by U the set of all points which are centres of a perspectivity mapping X_1 onto X_2. That is,

$$U = U(X_1,X_2) = \{x \in PG(2,q) \ : \ \pi_x^1(X_1) = X_2\}$$

$$= \{x \in PG(2,q) \ : \ \pi_x^2(X_2) = X_1\}.$$

Note that, if x, y are points in U, then $\pi_x^1 \pi_y^2$ induces an affinity π_{xy}^1 on the affine line $L_1 \setminus c$ and $\pi_x^2 \pi_y^1$ an affinity π_{xy}^2 on the affine line $L_2 \setminus c$. The affinities π_{xy}^1 and π_{xy}^2 preserve the sets X_1 and X_2, respectively. On the other hand, let x be in U, let φ_1 (resp. φ_2) be an affinity of $L_1 \setminus c$ (resp. $L_2 \setminus c$) preserving X_1 (resp. X_2). Assume $|U| \geq 2$. Then there exists a unique point $y \in U$ such that $\varphi_1 = \pi_{xy}^1$ (resp. $\varphi_2 = \pi_{xy}^2$). This can be shown using the fact that a projectivity fixing three points on a projective line is necessarily the identity on that line.

Fix any point $x \in U$. It then follows that

$$G_i = \{\pi_{xy}^i \quad : \quad y \in U\}\,,\ i = 1, 2,$$

are isomorphic groups; more precisely,

$$G_2 = \pi_x^2 G_1 \pi_x^1.$$

Also, for $i = 1, 2$, G_i is the full group of affinities of $L_i \setminus c$ preserving the set X_i. Furthermore,

- X_i is a union of orbits of the group G_i on $L_i \setminus c$, $i = 1, 2$;

- $|U| = |G_1| = |G_2|$.

We now recall that every subgroup G of the full affine group Σ has size tp^h. This can be seen from the fact that the full affine group has order $q(q-1)$. Some of these subgroups have the following structure:

$$G = G(A, B) = \{g \quad : \quad g(y) = ay + b\ ,\ a \in A\ ,\ b \in B\} \tag{1}$$

where

(i) B is a subspace of $GF(q)$ of dimension h_1 considered as a vector space over a subfield $GF(q_1)$ of $GF(q)$ with $q_1 = p^d$ and $d|r$. This implies that B is an additive subgroup of $GF(q)$ of order p^h with $h = dh_1$;

(ii) A is a multiplicative subgroup of $GF(q_1)$ of order t, where $t|(p^d - 1)$. In this way, B is invariant under A, i.e. $AB = B$.

We remark that, for every two integers $t|(p^d - 1)$ and $d|gcd(r, h)$, there exists in Σ a subgroup of type $G = G(A, B)$ of order tp^h, where B and A are additive and multiplicative subgroups of $GF(q)$ of order p^h and t, respectively.

It is not difficult to verify that a group G of type (1) has one orbit of length p^h on $AG(1, q)$, namely B, and that G acts regularly on the remaining orbits, say O_1,O_2,.. .,O_m, where

$$m = \frac{q - p^h}{tp^h} = \frac{p^{r-h} - 1}{t}.$$

Then

$$O_i = \{ay_i + b \quad : \quad a \in A \quad , \quad b \in B\}$$

where y_i is a suitable element of $GF(q) \setminus B$, $i = 1, 2, \ldots, m$, and

$$|O_i| = |G| = tp^h.$$

We point out the following.

(iii) The translations contained in G form a subgroup T of order p^h, namely

$$T = \{s \quad : \quad s(y) = y + b\,,\ b \in B\}.$$

(iv) Every element in $G \setminus T$ is a dilatation, *i.e.* has a unique fixed point.

(v) If $\varphi \in G \setminus T$, then the unique fixed point of φ on $AG(1, q)$ is an element of B.

(vi) The stabilizer G_b in G of any point $b \in B$ is a subgroup of dilatations of B and contains exactly t elements. More precisely,

$$G_b = \{g \quad : \quad g(y) = ay + (b - ab)\,,\ a \in A\}.$$

Finally we recall the following classification of the subgroups of Σ (*Chapter XII* of [4]).

Result 2.1 *Let Γ be a subgroup of Σ. If Γ is not of type (1), then it is conjugate to a subgroup of type (1) under a suitable non-trivial translation.*

Thus result 2.1 implies that every element in a subgroup G of Γ is of the form

$$x \rightarrow ax - u(a-1) + b$$

for a in A, b in B and u a fixed element of $GF(q)$. Using the change of coordinates given by $x' = x + u$ we may now assume that G is exactly as in (1).

We fix in $PG(2, q)$ a coordinate frame as follows. The lines L_1, L_2 have equations $x_2 = 0$, $x_3 = 0$, respectively. Set $x = (0, -1, 1)$. Put $O_1 = (0, 0, 1)$, $\pi^1_x(O_1) = (0, 1, 0)$, $U_1 = (1, 0, 1)$ and $\pi^1_x(U_1) = (1, 1, 0)$. Then the orbits of G_1 on the line $L_1 \setminus c$ are the following:

$$B^1 = \{(b, 0, 1) \,:\, b \in B\} \text{ and } O^1_i = \{(ac_i + b, 0, 1) \,:\, a \in A\,,\, b \in B\},$$

where c_i is a suitable element of $GF(q) \setminus B$, $i = 1, 2, \ldots, m$.

Since $\pi^1_x(x_1, 0, 1) = (x_1, 1, 0)$, for $x_1 \in GF(q)$, we obtain the orbits of G_2 on $L_2 \setminus c$, namely

$$B^2 = \{(b, 1, 0) \,:\, b \in B\} \text{ and } O^2_i = \{(ac_i + b, 1, 0) \,:\, a \in A\,,\, b \in B\}.$$

Then, since $|U| = |G_1| = |G_2|$, it is easy to check that

$$U = \{(b, -a, 1) \,:\, a \in A\,,\, b \in B\}.$$

Our discussion may be summarized as follows.

Result 2.2 *Let $X_1 \subseteq L_1 \setminus c$, $X_2 \subseteq L_2 \setminus c$ be two perspective sets in $PG(2, q)$. Then, using a suitable projective frame in $PG(2, q)$, there exist an additive subgroup B of $GF(q)$, as in (i), and a multiplicative subgroup A of $GF(q)$, as in (ii), such that*

$$G = G(A, B) \simeq G_1 \simeq G_2.$$

G_i is the full group of affinities of $L_i \setminus c$ preserving the set X_i, $i = 1, 2$. Moreover, X_i is a union of orbits of G_i on $L_i \setminus c$, $i = 1, 2$, and

$$|U| = |G| = tp^h.$$

In the sequel we denote by B^i the orbit of G_i on $L_i \setminus c$ corresponding to B and by $O^i_1, O^i_2, \ldots, O^i_m$ the remaining orbits, for $i = 1, 2$.

We remark that, with the notation of Result 2.2, if $x, y \in U$, then π^1_{xy} (resp. π^2_{xy}) is a translation of $L_1 \setminus c$ (resp. $L_2 \setminus c$) iff x, y, c are collinear and is a dilatation iff x, y, c are not collinear. Moreover, if x, y, c are collinear and we take a translation $\tau \in G_1$, $\tau \neq \pi^1_{xy}$, then, since τ has no fixed points on $L_1 \setminus c$, the unique point z such that $\tau = \pi^1_{xz}$ must be collinear with x and c. As G_1 contains exactly p^h translations, the line xy meets U in exactly p^h points. It follows that, if a line M through c intersects U, then $|U \cap M| = p^h$. On the other hand, if M misses c and $|M \cap U| \geq 2$, then from (vi) we have $|U \cap M| = t$. We also observe that, if two points $x, y \in U$ are on a line M not through c, then $\pi^1_{xy} \in G_1$ and $\pi^2_{xy} \in G_2$ are dilatations, so their fixed points are on B^1 and B^2, respectively, and such points are collinear with both x and y. We emphasize this fact as follows.

Result 2.3 *If a line M not through c meets U in at least two points, then M intersects both B^1 and B^2.*

Using the notation of 2.2 we can have, using 2.1 the following.

Result 2.4 *Exactly one of the following cases must occur:*

(j) *Both A and B are trivial. Then U consists of a singleton.*

(jj) *A is trivial and B is not trivial. Then U is a set of p^h points all collinear with the point c.*

(jjj) *B is trivial and A is not trivial. Then U is a set of t points on a line not through c.*

(jv) *A and B are the multiplicative and the additive group, respectively, of a subfield $GF(p^h)$ of $GF(q)$. Then*

$$U \cup B^1 \cup B^2 \cup \{c\} = PG(2, p^h).$$

(v) *None of the previous cases occur. Then U is a set of size tp^h and of type $[0, 1, t, p^h]$, i.e. $0, 1, t, p^h$ are the only intersection numbers of U with respect to the lines in $PG(2, q)$. In addition, using the fact that $|U| = tp^h$,*

(v,1) *there are exactly t lines intersecting U in exactly p^h points and they are all concurrent at the common point c of L_1 and L_2,*

(v,2) *each line intersecting U in exactly t points meets both B^1 and B^2.*

3. Examples of blocking sets with two Rédei lines

Using the previous notation, let $G = G(A, B)$ be a subgroup of Σ of order tp^h, $t = |A|$, $p^h = |B|$, with G acting on the affine line $L_1 \setminus c$. Denote by B^1 the orbit of G on $L_1 \setminus c$ corresponding to B and by $O^1_1, O^1_2, \ldots, O^1_m$ the remaining orbits. Fix a point $x \notin L_1 \cup L_2$ and consider on L_2 the point sets $B^2, O^2_1, \ldots, O^2_m$ defined by

$$\pi^1_x(B^1) = B^2 \ ; \ \pi^1_x(O^1_i) = O^2_i \, , \, i = 1, 2, \ldots, m.$$

Under this assumption, we also have

$$\pi^2_x(B^2) = B^1 \ ; \ \pi^2_x(O^2_i) = O^1_i \, , \, i = 1, 2, \ldots, m.$$

Recall that, for every $\varphi \in G$, there exists a unique point $y \notin L_1 \cup L_2$ such that $\varphi = \pi^1_{xy}$; so the set

$$U = U_x(G) = \{y \in PG(2, q) \ : \ \pi^1_{xy} \in G\}$$

contains exactly tp^h points and

$$G \simeq G_1 = \{\pi^1_{xy} \ : \ y \in U\} \simeq G_2 = \{\pi^2_{xy} \ : \ y \in U\}.$$

Of course, for $i = 1, 2$, the point sets $B^i, O^i_1, \ldots, O^i_m$ are the orbits of G_i on $L_i \setminus c$.

Next we consider two cases.

Example 3.1 Choose an arbitrary orbit O^1_i of G on L_1 and the corresponding one O^2_i on L_2. Define

$$X_1 = B^1 \cup O^1_1 \cup O^1_2 \cup \ldots \cup O^1_{i-1} \cup O^1_{i+1} \cup \ldots \cup O^1_m,$$

$$X_2 = B^2 \cup O^2_1 \cup O^2_2 \cup \ldots \cup O^2_{i-1} \cup O^2_{i+1} \cup \ldots \cup O^2_m$$

and put

$$Y = U \cup X_1 \cup X_2 \cup \{c\}.$$

Then, using Result 2.3, it is easy to verify that Y is a blocking set of size $q + \lambda$, with

$$\lambda = q - tp^h + 1,$$

and that L_1, L_2 are two Rédei lines of Y.

We point out that the five possibilities of Result 2.4 yield the following cases, respectively:

(j) G is trivial, $|U| = 1$ and $\lambda = q$.

(jj) A is trivial and B is not trivial. Then all orbits of G have length p^h, $|U| = p^h$ and $\lambda = q - p^h + 1$. Moreover, all points of U are on a line through the point c. So Y *is contained in three concurrent lines.*

(jjj) B is trivial and A is not trivial. Then G has one fixed point on both L_1 and L_2 (namely the unique element of B^1 and B^2, respectively) and m orbits each of size t. It follows that $|U| = t$ and $\lambda = q - t + 1$. Moreover, all points of U are on a line not through the point c. So Y *is contained in three non-concurrent lines.* We remark that in the case q odd and $t = \frac{q-1}{2}$ we obtain the so-called *projective triangle* ([9],[7],[3]).

(jv) A and B are the multiplicative and the additive group, respectively, of a subfield $GF(p^h)$ of $GF(q)$. In this case $\lambda = q - (p^h - 1)p^h + 1$ and $U = PG(2, p^h) \setminus (L_1 \cup L_2)$. We remark that if q is a square and $p^h = \sqrt{q}$, then Y is a *Baer subplane* of $PG(2, q)$.

(v) If none of the previous cases occurs, then U is a set of size tp^h and of type $[0, 1, t, p^h]$. In fact, the number of p^h-secants to U is exactly t, all of these pass through the point c and every secant not through c contains t points of U.

Example 3.2 Assume that neither A nor B is trivial. Define

$$X_1 = O_1^1 \cup O_2^1 \cup \ldots \cup O_m^1,$$

$$X_2 = O_1^2 \cup O_2^2 \cup \ldots \cup O_m^2.$$

Moreover, consider a subset X of U of size p^h consisting of p^h points collinear with c. If we put

$$Y = X \cup X_1 \cup X_2 \cup \{c\},$$

then it is easy to verify that Y is a blocking set of size $q + \lambda$, with

$$\lambda = q - p^h + 1,$$

and L_1, L_2 are two Rédei lines of Y.

We remark that, if we consider the above construction in the case that A is trivial, we obtain *(jj)* of example 3.1.

4. A characterization

Let Y be a Rédei blocking set of size $q + \lambda$ in $PG(2, q)$, with $q = p^r$, and assume that L_1, L_2 are two Rédei lines of Y. If we put

$$X = Y \setminus (L_1 \cup L_2) \quad ; \quad X_i = (Y \cap L_i) \setminus \{c\} \quad , \quad i = 1, 2,$$

then Y is partitioned into four disjoint sets, namely

$$Y = X \cup X_1 \cup X_2 \cup \{c\}$$

with

$$|X| = q - (\lambda - 1) \quad , \quad |X_1| = |X_2| = \lambda - 1.$$

We have that $\pi_x^1(X_1) = X_2$ and $\pi_x^2(X_2) = X_1$, for every $x \in X$. So the point sets X_1 , X_2 are perspective and, if

$$U = \{x \in PG(2, q) \, : \, \pi_x^1(X_1) = X_2\} = \{x \in PG(2, q) \, : \, \pi_x^2(X_2) = X_1\},$$

then

$$X \subseteq U.$$

Moreover, by result 2.2, we can choose a suitable coordinate system in such a way there exists a group $G = G(A, B)$ such that

$$G \simeq G_1 = \{\pi^1_{xy} \,:\, y \in U\} \simeq G_2 = \{\pi^2_{xy} \,:\, y \in U\},$$

where x is a fixed point in U. We have that $|B| = p^h$ and $|U| = tp^h$ using the notation of previous sections.

Now we consider two cases.

Case 1. Assume that B^1 is contained in X_1 and, as a consequence, that $B^2 \subseteq X_2$. Because X_1 is a union of G_1-orbits, then λ is at most $1 + q - tp^h$, since $\lambda < q$. The number of points of Y off L_1 is then at most $q - tp^h + |X|$. Moreover, this number must equal q. It follows that $|X| = tp^h$. Since $X \subseteq U$ and $|U| = tp^h$, we have $X = U$. We conclude that Y is a blocking set of the type described in Example 3.1 of Sect.3.

Case 2. Assume that $B_1 \subseteq L_1 \backslash X_1$ and, as a consequence, that $B_2 \subseteq L_2 \backslash X_2$. Then λ is at most $1 + q - p^h$, since $\lambda < q$. So $|X| \geq p^h$. But, as in paper, since the join of any two points of X meets c, we have $|X| \leq p^h$. Thus $|X| = p^h$. Now we point out that, since L_1,L_2 are Rédei lines of Y, the line joining two distinct points x, y of X meets $X_1 \cup X_2 \cup \{c\}$. On the other hand, by Result 2.3, a line through two points of X not collinear with c must intersect both B^1 and B^2. Now $B^1 \not\subseteq X_1$, $B^2 \not\subseteq X_2$. It follows that x, y, c are collinear and, as a consequence, that X consists of p^h points of U collinear with c.
We conclude that Y is a blocking set of the type described in Example 3.2 of Sect.3.

Finally, we can state the following version of Sherman's result.

Theorem 4.1 *Let Y be a blocking set in $PG(2, q)$ having at least two Rédei lines. Then, with respect to a suitable projective coordinate system, Y is one of the two types described in Sect.3.*

By Theorem 4.1 and looking at the list of examples in Sect.3, all blocking sets of $PG(2, q)$ containing at least three Rédei lines can be easily classified.

Theorem 4.2 *Let Y be a blocking set in $PG(2, q)$ of size $q + \lambda$ having at least three Rédei lines. Then one of the following possibilities can occur:*

(1) *q is odd, $\lambda = \frac{q+3}{2}$ and Y is a projective triangle. In this case Y is contained in the union of three non concurrent Rédei lines.*

(2) *$q = p^{d+h}$, $t = p^d - 1$, $\lambda = p^h + 1$ and Y is the set $U \cup B^1 \cup B^2 \cup \{c\}$. In this case every line through c, which meets Y in at least two points, is a Rédei line.*

5. Transversal designs of index 1 and their embeddings

Recall [5] that a transversal design of order n, block size k and index λ, denoted $TD_\lambda(k,n)$, is a triple (V,S,B) where

(1) V is a set of kn elements;

(2) S is a partition of V into k classes, called *groups*, each of size n;

(3) B is a collection of $k-$subsets of V, called *blocks* ;

(4) every unordered pair of elements from V is either contained in exactly one group, or is contained in exactly λ blocks, but not both.

When $\lambda = 1$ one writes simply $TD(k,n)$. These transversal designs have a long history because, for example, a $TD(k,n)$ is equivalent to a set of $k-2$ mutually orthogonal latin squares of side n and also to an orthogonal array of strength two having n^2 columns, k rows and n symbols. A $TD(k,n)$ is said to be *embedded* in $PG(2,q)$ if there exist k lines in $PG(2,q)$ each containing a group of $TD(k,n)$. Here we provide a complete answer to the question: *which finite systems $TD(k,n)$ are embedded in $PG(2,q)$*? This answer sharpens the result in [5] for finite fields. Actually, taking into account the Theorem 4.1 of [5] and the two cases examined in its proof, it is not so difficult to prove the following structure theorem for a $TD(k,n)$ embedded in $PG(2,q)$.

Theorem 5.1 *Let Ω be a $TD(k,n)$ with k and n at least 3 which is embedded in $PG(2,q)$. Then coordinates may be chosen such that the points of Ω and the lines of Ω are subsets of the points and lines of one of the following examples:*

(a) An example modelled on case (jjj) of Result 2.4. The point set of the design is $U \cup O^1 \cup O^2$, where

$$O^1 = \{(a,0,1) \ : \ a \in A\} \ , \quad O^2 = \{(a,1,0) \ : \ a \in A\} \ ,$$

and each line contains exactly $t = |A|$ points. In the q odd case, if t divides $\frac{q-1}{2}$, then Ω is embedded in a projective triangle.

(b) A subplane as in case (jv) of Result 2.4. The points of the design are all points of the subplane apart from a single point c which is removed. The groups are points of the subplane on lines through c.

(c) *An example modelled on case (v) of Result 2.4. The point set of the design is* $U \cup B^1 \cup B^2$ *and each group contains exactly* $|B| = p^h$ *points lying on a line through c. This example also includes the case when the TD has just 3 lines as in case (jj) of Result 2.4.*

FINAL REMARK Many of our results can be extended to higher dimensions. We will report on this elsewhere.

ACKNOWLEDGMENTS The first author wishes to thank the *National Science and Engineering Research Council of Canada* (NSERC) for its support. The second and the third author wish to thank for their supports the research group *GNSAGA* of Italian *Istituto Nazionale di Alta Matematica* and the *Dipartimento di Matematica* of the *Seconda Università degli Studi di Napoli.*

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